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# A local version of the Pawłucki–Pleśniak extension operator

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**Abstract**

Using local interpolation of Whitney functions, we generalize the Pawłucki and Pleśniak approach to construct a continuous linear extension operator. We show the continuity of the modified operator in the case of generalized Cantor-type sets without Markov's Property.


*MSC:* primary 46E10; secondary 41A05; 41A10

*Keywords:* Whitney functions; Extension operator; Local interpolation; Cantor-type sets; Markov's Property

## 1. Introduction

For a compact set $K \subset \mathbb{R}^d$, let $\mathcal{E}(K)$ denote the space of Whitney jets on $K$ (see e.g. [24] or [11]). The problem of the existence of an extension operator (here and in what follows it means a continuous linear extension operator) $L : \mathcal{E}(K) \longrightarrow C^\infty(\mathbb{R}^d)$ was first considered in [4,13,20,21]. In [22], a topological characterization (*DN* property) for the existence of an extension operator was given. In elaboration of Whitney's method Schmets and Valdivia proved in [19] (see also [7]) that if the extension operator $L$ exists, then one can take a map such that all extensions are analytic on the complement of the compact set. For the extension problem in the classes of ultradifferentiable functions see, for example, [5,17]

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and the references therein. In [14] (see also [15,18]), Pawłucki and Pleśniak suggested an explicit construction of the extension operator for a rather wide class of compact sets, preserving Markov's inequality. In [8] and later in [9], the compact sets $K$ were presented without Markov's Property, such that the space $\mathcal{E}(K)$ admitted an extension operator. Here, we deal with the generalized Cantor-type sets $K^{(\alpha)}$ that have the extension property for $1 < \alpha < 2$, as it was proved in [9], but are not Markov's sets for any $\alpha > 1$ in accordance with Pleśniak's [16] and Białas's [3] results. The extension operator in [14] was given in the form of a telescoping series containing Lagrange interpolation polynomials with the Fekete–Leja system of knots. This operator is continuous in the Jackson topology $\tau_J$, which is equivalent to the natural topology $\tau$ of the space $\mathcal{E}(K)$, provided that the compact set $K$ admits Markov's inequality. Here, following [10], we interpolate the functions from $\mathcal{E}(K^{(\alpha)})$ locally and show that the modified operator is continuous in $\tau$.

## 2. Jackson topology

For a perfect compact set $K$ on the line, $\mathcal{E}(K)$ denotes the space of all functions $f$ on $K$ extendable to some $F \in C^\infty(\mathbb{R})$. The topology $\tau$ of Fréchet space in $\mathcal{E}(K)$ is given by the norms

$$\| f \|_q = |f|_q + \sup\{ |(R_y^q f)^{(k)}(x)| \cdot |x-y|^{k-q};\ x, y \in K, x \neq y,\ k = 0, 1, \ldots, q\},$$

$q = 0, 1, \ldots,$ where $|f|_q = \sup\{|f^{(k)}(x)| : x \in K, k \leqslant q\}$ and $R_y^q f(x) = f(x) - T_y^q f(x)$ is the Taylor remainder.

The space $\mathcal{E}(K)$ can be identified with the quotient space $C^\infty(I)/Z$, where $I$ is a closed interval containing $K$ and $Z = \{F \in C^\infty(I) : F|_K \equiv 0\}$. Given $f \in \mathcal{E}(K)$, let $||| f |||_q = \inf |F|_q^{(I)}$, where the infimum is taken for all possible extensions of $f$ to $F$ and $|F|_q^{(I)}$ denotes the $q$th norm of $F$ in $C^\infty(I)$. The quotient topology $\tau_Q$, given by the norms $(||| \cdot |||_q)$, is complete; by the open mapping theorem, it is equivalent to the topology $\tau$. Therefore, for any $q$ there exists $r \in \mathbb{N},\ C > 0$ such that

$$||| f |||_q \leqslant C \, || f ||_r \tag{1}$$

for any $f \in \mathcal{E}(K)$.

Following Zerner [25], Pleśniak [15] introduced in $\mathcal{E}(K)$ the following seminorms:

$$d_{-1}(f) = |f|_0,\ d_0(f) = E_0(f),\ d_k(f) = \sup_{n \geqslant 1} n^k E_n(f)$$

for $k = 1, 2, \ldots$ . Here, $E_n(f)$ denotes the best approximation to $f$ on $K$ by polynomials of degree at most $n$. For a perfect set $K \subset \mathbb{R}$ the *Jackson topology* $\tau_J$, given by $(d_k)$, is Hausdorff. By the Jackson theorem (see, e.g. [23]) the topology $\tau_J$ is well-defined and is not stronger than $\tau$.

The characterization of analytic functions on a compact set $K$ in terms of $(d_k)$ was considered in [2]; for the spaces of ultradifferentiable functions see [6].

We remark that for any perfect set $K$, the space $(\mathcal{E}(K), \tau_J)$ has the dominating norm property (see, e.g. [12]):

$$\exists p \ \forall q \ \exists r, \ C > 0: \ d_q^2(f) \leqslant C \ d_p(f)\, d_r(f) \quad \text{for all} \quad f \in \mathcal{E}(K).$$

Indeed, let $n_k$ be such that $d_k(f) = n_k^k \, E_{n_k}(f)$. Then, $d_p(f) \geqslant n_q^p \, E_{n_q}(f)$ and $d_r(f) \geqslant n_q^r$ $E_{n_q}(f)$. So we have the desired condition with $r = 2\,q$.

Tidten proved in [22] that the space $\mathcal{E}(K)$ admits an extension operator if and only if it has the property $(DN)$. Clearly, the completion of the space with the property $(DN)$ also has the dominating norm. Therefore, the Jackson topology is not generally complete. Moreover, it is not complete in the cases of compact sets from [8,9] in spite of the fact that the corresponding spaces have the $(DN)$ property. By Theorem 3.3 in [15], the topologies $\tau$ and $\tau_J$ coincide for $\mathcal{E}(K)$ if and only if the compact set $K$ satisfies the Markov Property (see [14–18] for the definition) and this is possible if and only if the extension operator, presented in [14,15,18], is continuous in $\tau_J$. We do not know the distribution of the Fekete points for Cantor-type sets, and therefore we cannot check the continuity of the Pawłucki and Pleśniak operator in the natural topology. Instead, following [10], we will interpolate the functions from $\mathcal{E}(K)$ locally.

## 3. Extension operator for $\mathcal{E}(K^{(\alpha)})$

Let $(l_s)_{s=0}^{\infty}$ be a sequence such that $l_0 = 1, 0 < 2l_{s+1} < l_s$, $s \in \mathbb{N}$. Let $K$ be the Cantor set associated with the sequence $(l_s)$, that is, $K = \bigcap_{s=0}^{\infty} E_s$, where $E_0 = I_{1,0} = [0, 1]$, $E_s$ is a union of $2^s$ closed *basic* intervals $I_{j,s}$ of length $l_s$ and $E_{s+1}$ is obtained by deleting the open concentric subinterval of length $l_s - 2l_{s+1}$ from each $I_{j,s}$ , $j = 1, 2, \ldots, 2^s$.

Fix $1 < \alpha < 2$ and $l_1$ with $2l_1^{\alpha-1} < 1$. We will denote by $K^{(\alpha)}$ the Cantor set associated with the sequence $(l_n)$, where $l_0 = 1$ and $l_{n+1} = l_n^{\alpha} = \cdots = l_1^{\alpha^n}$ for $n \geqslant 1$.

In the notations of Arslan et al. [1], we consider the set $K_2^{(\alpha)}$. The construction of the extension operator for the case $K_n^{(\alpha)}$ with $\alpha < n$ is quite similar, so we can restrict ourselves to $n = 2$.

Let us fix $s, \ m \in \mathbb{N}$ and take $N = 2^m - 1$. The interval $I_{1,s}$ covers $2^{m-1}$ basic intervals of the length $l_{s+m-1}$. Then $N+1$ endpoints $(x_k)$ of these intervals give us the interpolating set of the Lagrange interpolation polynomial $L_N(f, x, I_{1,s}) = \sum_{k=1}^{N+1} \ f(x_k)\, \omega_k(x)$, corresponding to the interval $I_{1,s}$. Here, $\omega_k(x) = \frac{\Omega_{N+1}(x)}{(x-x_k)\Omega'_{N+1}(x_k)}$ with $\Omega_{N+1}(x) = \Pi_{k=1}^{N+1}(x-x_k)$. In the case $2^m < N+1 < 2^{m+1}$, we use the same procedure as in [10] to include new $N+1-2^m$ endpoints of the basic intervals of the length $l_{s+m}$ in the interpolation set. The polynomials $L_N(f, x, I_{j,s})$, corresponding to other basic intervals, are taken in the same manner.

Given $\delta > 0$, and a compact set $E$, we take a $C^{\infty}$-function $u(\cdot, \delta, E)$ with the properties: $u(\cdot, \delta, E) \equiv 1$ on $E$, $u(x, \delta, E) = 0$ for $\text{dist}(x, E) > \delta$ and $|u|_p \leqslant c_p \ \delta^{-p}$, where the constant $c_p$ depends only on $p$. Let $(c_p) \uparrow$ .

Fix $n_s = [s \log_2 \alpha]$ for $s \geqslant \log 4/\log \alpha$, $n_s = 2$ for the previous values of $s$ and $\delta_{N,\,s} = l_{s+[\log_2 N]}$ for $N \geqslant 2$. Here $[a]$ denotes the greatest integer in $a$.

Let $N_s = 2^{n_s} - 1$ and $M_s = 2^{n_{s-1}-1} - 1$ for $s \geqslant 1$, $M_0 = 1$. Consider the operator from [10]

$$
\begin{aligned}
L(f,x) = {} & L_{M_0}(f,x,I_{1,0})\,u(x,\delta_{M_0+1,0},I_{1,0}\cap K) \\
& + \sum_{s=0}^{\infty}\left\langle \sum_{j=1}^{2^s}\sum_{N=M_s+1}^{N_s}[L_N(f,x,I_{j,s}) - L_{N-1}(f,x,I_{j,s})] \right. \\
& \times u(x,\delta_{N,s},I_{j,s}\cap K) \\
& + \sum_{j=1}^{2^{s+1}}[L_{M_{s+1}}(f,x,I_{j,s+1}) - L_{N_s}(f,x,I_{[\frac{j+1}{2}],s})] \\
& \left. \times\ u(x,\delta_{N_s,s},I_{j,s+1}\cap K)\right\rangle.
\end{aligned}
$$

We call the sums $\sum_{N=M_s+1}^{N_s}\cdots$ the *accumulation sums*. For fixed $j$ (without loss of generality let $j = 1$) represent the term in the last sum in the telescoping form

$$
-\sum_{N=2^{n_s-1}}^{2^{n_s}-1}[L_N(f,x,I_{1,s}) - L_{N-1}(f,x,I_{1,s})]\,u(x,l_{s+n_s-1},I_{1,s+1}\cap K)
$$

and will call this the *transition sum*. Here, the interpolation set for the polynomial $L_N(f,x,I_{1,s})$ consists of all endpoints of the basic subintervals of length $l_{s+n_s-1}$ on $I_{1,s+1}$ and some endpoints (from 0 for $N = 2^{n_s-1} - 1$ to all for $N = 2^{n_s} - 1$) of basic subintervals of the same length on $I_{2,s+1}$.

Clearly, the operator $L$ is linear. Let us show that it extends the functions from $\mathcal{E}(K^{(\alpha)})$.

**Lemma 1.** *For any* $f \in \mathcal{E}(K^{(\alpha)})$ *and* $x \in K^{(\alpha)}$, *we have* $L(f,x) = f(x)$.

**Proof.** By the telescoping effect

$$
L(f,x) = \lim_{s\to\infty} L_{M_s}(f,x,I_{j,s}), \tag{2}
$$

where $j = j(s)$ is chosen in such a way that $x \in I_{j,s}$.

We will denote temporarily $n_{s-1} - 1$ by $n$. Then $M_s = 2^n - 1$. Arguing as in [10], for any $q$, $1 \leqslant q \leqslant M_s$, we have the bound

$$
|\,L_{M_s}(f,x,I_{j,s}) - f(x)| \leqslant ||\,f\,||_q \sum_{k=1}^{2^n} |\,x - x_k|^q\,|\,\omega_k(x)\,|. \tag{3}
$$

For the denominator of $|\,\omega_k(x)\,|$ we get

$$
\begin{aligned}
& |\,x_k - x_1|\cdots|\,x_k - x_{k-1}|\cdot|\,x_k - x_{k+1}|\cdots|\,x_k - x_{M_s+1}| \\
& \quad \geqslant l_{n+s-1}\,(l_{n+s-2} - 2\,l_{n+s-1})^2\cdot(l_{n+s-3} - 2\,l_{n+s-2})^4\cdots(l_s - 2\,l_{s+1})^{2^{n-1}} \\
& \quad = l_{n+s-1}\cdot l_{n+s-2}^2\cdots l_s^{2^{n-1}}\cdot A,
\end{aligned}
$$

where $A = \Pi_{k=1}^{n-1}\,(1 - 2\,\frac{l_{s+k}}{l_{s+k-1}})^{2^{n-k}}$.

Clearly, $\ln\, A > -\sum_{k=1}^{n-1} 2^{n-k+2} \frac{l_{s+k}}{l_{s+k-1}}$ for large enough $s$. Since $\frac{l_{s+k}}{l_{s+k-1}} < \frac{l_{s+k-1}}{l_{s+k-2}}$ and $2^n \leqslant \frac{1}{2}\, \alpha^{s-1}$ , we have $\ln\, A > -2^{n+2}\, l_s^{\alpha-1} > -1$.

On the other hand, the numerator of $|\, \omega_k(x)\, |$ multiplied by $|\, x - x_k|^{\, q}$ gives the bound

$$|\, x - x_k|^{\, q-1}\, \Pi_1^{2^n} |\, x - x_k| \leqslant l_s^{q-1} \cdot l_{n+s} \cdot l_{n+s-1} \cdot l_{n+s-2}^2 \cdots l_s^{2^{n-1}}.$$

Hence, the sum in (3) may be estimated from above by $e\ 2^n\ l_{n+s}\, l_s^{q-1}$, which approaches 0 as $s$ becomes large. Therefore, the limit in (2) exists and equals $f(x)$. □

## 4. Continuity of the operator L

**Theorem 1.** *Let* $1 < \alpha < 2$. *The operator* $L : \mathcal{E}(K^{(\alpha)}) \longrightarrow C^\infty(\mathbb{R})$, *given in Section* 3, *is a continuous linear extension operator.*

**Proof.** Let us prove that the series representing the operator $L$ uniformly converges together with any of its derivatives.

For any $p \in \mathbb{N}$, let $q = 2^v - 1$ be such that $(2/\alpha)^v > p + 4$. Given $q$ let $s_0$ satisfy the following conditions: $s_0 \geqslant 2v + 3$ and $\alpha^m \geqslant m$ for $m \geqslant n_{s_0-1}$.

Suppose the points $(x_k)_1^{N+1}$ are arranged in ascending order. For the divided difference $[x_1, \ldots, x_{N+1}]f$, we have the following bound from [10]:

$$|\ [x_1, \ldots, x_{N+1}]f\ | \leqslant 2^{N-q}\ |||f|||_q\ \ (\min\ \Pi_{m=1}^{N-q} |\, x_{a(m)} - x_{b(m)}|)^{-1}, \tag{4}$$

where min is taken over all $1 \leqslant j \leqslant N + 1 - q$ and all possible chains of strict embeddings $[x_{a(0)}, \ldots, x_{b(0)}] \subset [x_{a(1)}, \ldots, x_{b(1)}] \subset \cdots \subset [x_{a(N-q)}, \ldots, x_{b(N-q)}]$ with $a(0) = j,\ b(0) = j + q, \ldots, a(N - q) = 1,\ b(N - q) = N + 1$. Here, given $a(k),\ b(k)$, we take $a(k + 1) = a(k),\ b(k + 1) = b(k) + 1$ or $a(k + 1) = a(k) - 1,\ b(k + 1) = b(k)$. The length of the first interval in the chain is not included in the product in (4), which we denote in the sequel by $\Pi$.

For $s \geqslant s_0$ and for any $j \leqslant 2^s$ we consider the corresponding term of the accumulation sum. By the Newton form of interpolation operator we get

$$L_N(f, x, I_{j,s}) - L_{N-1}(f, x, I_{j,s}) = [x_1, \ldots, x_{N+1}]f \cdot \Omega_N(x),$$

where $\Omega_N(x) = \Pi_1^N(x - y_k)$ with the set $(y_k)_1^N$ consisting of all points $(x_k)_1^{N+1}$ except one.

Thus, we need to estimate $|\ [x_1, \ldots, x_{N+1}]f\ |\ \cdot |(\,\Omega_N \cdot u(x, \delta_{N,\,s}, I_{j,s} \cap K))^{(p)}|$ from above. Here $M_s + 1 \leqslant N \leqslant N_s$, that is $2^{m-1} \leqslant N < 2^m$ for some $m = n_{s-1}, \ldots, n_s$ and $\delta_{N,\,s} = l_{s+m-1}$. The interpolation set $(x_k)_1^{N+1}$ consists of all endpoints of the basic intervals of length $l_{s+m-2}$ (inside the interval $I_{j,s}$) and some endpoints (possibly all for $N = 2^m - 1$) of the basic intervals of length $l_{s+m-1}$. For simplicity we take $j = 1$. In this case, $x_1 = 0,\ x_2 = l_{s+m-1},\ x_3 = l_{s+m-2} - l_{s+m-1}$ or $x_3 = l_{s+m-2}$, etc.

Since $\mathrm{dist}(x, I_{1,s} \cap K) \leqslant l_{s+m-1}$, we get

$$|\, \Omega_N^{(i)}(x)\, | \leqslant \frac{N!}{(N-i)!}\ \Pi_{k=i+1}^N (l_{s+m-1} + y_k).$$

Therefore, $|(\Omega_N \cdot u)^{(p)}| \leqslant \sum_{i=0}^{p} \binom{p}{i} c_{p-i}\, l_{s+m-1}^{i-p}\, N^i\, \Pi_{k=i+1}^{N}(l_{s+m-1}+y_k) \leqslant 2^p\, c_p\, l_{s+m-1}^{-p}\, \Pi_{k=1}^{N}(l_{s+m-1}+y_k).\ \max_{i\leqslant p} B_i$, with $B_0=1,\ B_1=N,\ B_2=N^2/2, \ldots, B_i=N^2/2\cdot (N\, l_{s+m-1})^{i-2}\ (l_{s+m-1}+y_3)^{-1}\cdots(l_{s+m-1}+y_i)^{-1}$ for $i\geqslant 3$.

To estimate $B_3$, we note that $l_{s+m-1}+y_3 \geqslant l_{s+m-2},\ N\, l_{s+m-1} < 2^m\, l_{s+m-2}^{\alpha} \leqslant l_{s+m-2}$ since $2^m\, l_{s+m-2}^{\alpha-1} = 2^m\, l_{s-2}^{(\alpha-1)\alpha^m} < 2^m\, l_1^{(\alpha-1)\alpha^m} < 2^m (\frac{1}{2})^{\alpha^m} \leqslant 1$, due to the choice of $s_0$. Therefore, $B_3$, and all $B_i$ for $i>3$, are less than $B_2$. On the other hand, $l_{s+m-1}+y_k < y_{k+1},\ k\leqslant N-1$, as $l_{s+m-1}$ is a mesh of the net $(y_k)_1^N$ and $l_{s+m-1}+y_N < 2l_s$. This implies that

$$|(\Omega_N \cdot u)^{(p)}| \leqslant 2^p\, c_p\, N^2\, l_{s+m-1}^{-p}\, l_s\, \Pi_{k=2}^{N} y_k \leqslant 2^p\, c_p\, N^2\, l_{s+m-1}^{-p-1}\, l_s\, \Pi_{k=2}^{N+1} x_k. \tag{5}$$

To apply (4), for $1\leqslant j\leqslant N+1-q$ we consider $q+1$ consecutive points $(x_{j+k})_{k=0}^{q}$ from $(x_k)_1^{N+1}$. Every interval of the length $l_{s+k}$ contains from $2^{m-k-1}+1$ to $2^{m-k}$ points $x_k$. Therefore, the interval of the length $l_{s+m-v-1}$ contains more than $q+1$ points. In order to minimize the product $\Pi$, we have to include intervals containing large gaps in the set $K^{(\alpha)}$ in the chain $[x_j,\ldots,x_{j+q}] \subset \cdots \subset [x_1,\ldots,x_{N+1}]$ as late as possible, that is all $q+1$ points must belong to $I_{j,s+m-v-1}$ for some $j$. By the symmetry of the set $K^{(\alpha)}$, we can again take $j=1$. The interval of the length $l_{s+m-v}$ contains at most $2^v$ points, whence for any choice of $q+1$ points in succession, all values that make up the product $\Pi$ are not smaller than the length of the gap $h_{s+m-v-1} := l_{s+m-v-1} - 2\, l_{s+m-v}$. Therefore, $\Pi \geqslant h_{s+m-v-1}^{J-q-1}\, \Pi_{J+1}^{N+1} x_k$, where $J$ is the number of points $x_k$ on $I_{1,s+m-v-1}$. Since $J\leqslant 2^{v+1}$, we have $J-q-1\leqslant 2^v$. Further,

$$\frac{x_{q+2}\cdots x_J}{h_{s+m-v-1}^{J-q-1}} \leqslant \left(\frac{l_{s+m-v-1}}{l_{s+m-v-1}-2\, l_{s+m-v}}\right)^{2^v} < \exp(2^v\, 4 l_{s+m-v-1}^{\alpha-1}). \tag{6}$$

Since $l_{s+m-v-1}^{\alpha-1} = l_1^{(\alpha-1)(s+m-v-2)} < 2^{-s+v}$, we see that the fraction above is smaller than 2, due to the choice of $s_0$. It follows that $\Pi \geqslant \frac{1}{2}\Pi_{q+2}^{N+1} x_k$ and $|[x_1,\ldots,x_{N+1}]f| \leqslant 2^{N-q-1}\ |||f|||_q\ (x_{q+2}\cdots x_{N+1})^{-1}$.

Combining this with (5) we have

$$|\ [x_1,\ldots,x_{N+1}]f\ |\ \cdot |(\Omega_N\cdot u)^{(p)}| \leqslant c_p\, N^2\, 2^N\, l_s\ l_{s+m-1}^{-p-1}\, \Pi_{k=2}^{q+1} x_k\, |||f|||_q.$$

Our next goal is to evaluate $\Pi_{k=2}^{q+1} x_k$ in terms of $l_{s+m-1}$. Estimating roughly all $x_k,\ k>2$ that are not endpoints of the basic intervals of length $l_{s+m-2}$, from above by $l_{s+m-v-1}$, we get

$$\Pi_{k=2}^{q+1} x_k \leqslant l_{s+m-1}\, l_{s+m-2}\, l_{s+m-3}^2 \cdots l_{s+m-v}^{2^{v-2}}\, l_{s+m-v-1}^{2^{v-1}-1} = l_{s+m-1}^{\kappa}$$

with $\kappa = 1+\frac{1}{\alpha}+\frac{2}{\alpha^2}+\cdots+\frac{2^{v-1}}{\alpha^v}-\frac{1}{\alpha^v} > (2/\alpha)^v - 1$.

Therefore,

$$|\ [x_1,\ldots,x_{N+1}]f\ |\ \cdot |(\Omega_N\cdot u)^{(p)}| \leqslant c_p\, N^2\, 2^N\ l_{s+m-1}^2\, |||f|||_q,$$

since $\kappa+\alpha^{-m+1}-p-1>2$, due to the choice of $q$.

Here, $2^N\ l_{s+m-1} < 2^{2^m}\ l_1^{\alpha^{s+m-2}} < 2^{2^{n_s}-\alpha^s} \leqslant 1$, as $m \geqslant 2$ and $l_1 < \frac{1}{2}$. The accumulation sum contains $N_s - M_s < N_s$ terms. Therefore,

$$\left| \left( \sum_{N=M_s+1}^{N_s} \cdots \right)^{(p)} \right| \leqslant c_p\, N_s^3\, l_s\, |||f|||_q,$$

which is a term of the series convergent with respect to $s$, as is easy to see. We neglect the sum with respect to $j$, because for fixed $x$, at most one term of this sum does not vanish.

The same proof works for the terms of the transition sums. This sum does not vanish only for $x$ at a short distance to $I_{1,s+1} \cap K$. For this reason, the arguments of the estimation of $|\,\Omega_N^{(i)}(x)|$ remain valid. On the other hand, if we want to minimize the product of the lengths of intervals, constituting the chain $[x_j, \ldots, x_{j+q}] \subset \cdots \subset [x_1, \ldots, x_{N+1}]$, then we have to take $x_j, \ldots, x_{j+q}$ in the interval $I_{1,s+1}$. Thus we have the bound (6). The rest of the proof runs as before. Taking into account (1), we see that the operator $L$ is well-defined and continuous. □

**Remark.** It is a simple matter to find a sequence of functions that converges in the Jackson topology and diverges in $\tau$. It is interesting that the same sequence can destroy the Markov inequality. Given $s \in \mathbb{N}$, let $N = 2^s$ and $P_N(x) = (l_{s-1} \cdot l_{s-2}^2 \cdots l_0^{2^{s-1}})^{-1} \prod_{j=1}^N (x - c_{j,s})$, where $c_{j,s}$ is a midpoint of the interval $I_{j,s}$. Then $\frac{1}{s}\ \ln(|P_N'(0)|/|P_N|_0) \to \infty$ as $s \to \infty$, contrary to the Markov property. On the other hand, $E_n(P_N) \leqslant |P_N|_0$ for $n < N$. Then, for any $k$ we get $d_k(P_N) \leqslant N^k\, |P_N|_0 \leqslant 2^{s\,k}\, l_s \to 0$ as $s \to \infty$. But $P_N'(0) \nrightarrow 0$, so the sequence $(P_N)$ diverges in the natural topology of the space $\mathcal{E}(K^{(\alpha)})$.

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